Pattern formation in an activator-inhibitor model: effect of albedo boundary conditions on a finite geometry

1993 J. Phys. A: Math. Gen. 26 4281

(http://iopscience.iop.org/0305-4470/26/17/033)

J. Phys. A: Math. Gen. 26 (1993) 4281–4286. Printed in the UK

Pattern formation in an activator–inhibitor model: Effect of albedo boundary conditions on a finite geometry

Horacio S Wio†, G Izus†, O Ramirez‡, R Deza‡ and Carlos Borzi§

† Centro Atómico Bariloche* and Instituto Balseiro 8400—San Carlos de Bariloche, Argentina

‡ Departamento de Física, Facultad de Ciencias Exactas y Naturales Universidad Nacional de Mar del Plata Funes 3350, 7600—Mar del Plata, Argentina

§ Gerencia Investigación y Desarrollo, Centro Atómico Ezeiza*, Buenos Aires, Argentina

Received 20 July 1992

Abstract. We study a piecewise linear version of an activator–inhibitor model with the aim of analysing the effect of albedo boundary conditions on the formation and stability of patterns. We find concentration profiles for both components and analyse the linear stability propperties of those profiles. We show that it is possible, under certain conditions, to control the shape and the stability of the patterns. Also, a scaling behaviour on the marginal stability line has been found.

Pattern formation and propagation is one of the main issues in the physics of complex systems [1]. Among the several frameworks in which these phenomena can be described, the reaction-diffusion approach has provided a very fertile source of models, for instance, in the description of chemical clocks (e.g. the Belousov–Zhabotinski reaction), the propagation of neural signals along the axonal membrane, temporally periodic or spatially organized activity in heart tissue, etc. [2–5]. In particular, the possibility of a solitary pattern without propagation in an infinite medium was established [6].

We shall investigate the effect of different boundary conditions on the possibility of existence and stability of patterns in systems defined on a finite one-dimensional space. For a one-component system, the boundary conditions have been shown recently to play a relevant role on the formation and stability of patterns [7]. In the two-component case to be analysed here we shall impose albedo boundary conditions (that is, a linear combination of Dirichlet and Neumann conditions) along the lines of a recently studied model for an electrothermal instability [8]. As was indicated in [8], the choice of Neumann or Dirichlet boundary conditions, physically corresponds to complete reflection or absorption at the boundary, respectively. The more realistic case of partially reflecting (or partially absorbing) boundaries, is adequately taken into account by the albedo boundary conditions. In the present analysis, we restrict ourselves to the symmetric case, that is when the albedo parameter is the same for both fields. We then perform a standard linear stability analysis which is implemented numerically. The specific model which we shall concentrate upon belongs to a family

* Comisíon Nacional de Energía Atómica.

of multicomponent models with a broad range of applicability [9]. This is the so-called *propagator–controller* or *activator–inhibitor* model, whose general formulation reads

$$\frac{\partial}{\partial t}X(z,t)=D_x\frac{\partial^2}{\partial z^2}X+F(X,Y) \tag{1}$$

$$\frac{\partial}{\partial t}Y(z,t)=D_y\frac{\partial^2}{\partial z^2}Y+G(X,Y)$$

where X and Y are the concentration fields of both species, D_x and D_y are their respective diffusion coefficients and $F(X,Y)$, $G(X,Y)$ represent the reaction sources. The nullclines, that is the intersections of those (generally nonlinear) source terms with the (X,Y) plane show characteristic shapes which can be described, typically, by a convex line for G and a general cubic-like one (with two extrema and one inflection point) for F [10]. Those projections intersect each other at the origin (which counts for a trivial solution) and eventually on both sides around the local maximum of F. Those extra intersections pre-announce non-trivial solutions of the system and have been analysed by several authors on a simplified (piecewise linear) version of the model [10]. Those authors imposed straightforward Neumann boundary conditions for the inifinite system and found stable solutions which behave, under certain conditions, as self-confined stationary patterns in an infinite system. We are interested in finite geometries where it is legitimate to ask for other (more general) types of boundary conditions, namely the albedo ones.

We start with the simplified version of the FitzHugh–Nagumo model alluded to above and fix the parameters so as to allow for non-trivial solutions to exist. After rescaling the fields in a standard manner [6], we get a dimensionless version of the model as

$$\frac{\partial}{\partial t}X(z,t)=D_x\frac{\partial^2}{\partial z^2}X-X-aY+\gamma\theta(X-X_c)$$

$$\frac{\partial}{\partial t}Y(z,t)=D_y\frac{\partial^2}{\partial z^2}Y+bX-cY. \tag{2}$$

We confine the system to the interval $-z_L<z<z_L$ and impose the albedo boundary conditions as

$$\left.\frac{dX}{dz}\right|_{z=\pm z_L}\mp k_xX(\mp z_L)=0 \qquad \left.\frac{dY}{dZ}\right|_{z=\pm z_L}\mp k_yY(\pm z_L)=0. \tag{3}$$

Note that, in the symmetric case, we are considering here, there is a unique albedo parameter which varies from 0 to ∞ (these limits correspond to pure Neumann and pure Dirichlet boundary conditions, respectively). We shall work on a dimensionless spatial coordinate, scaled with the size of the system (z/L) and, as we are going to propose spatially even solutions, we shall only study positive values of z_L. As was already discussed for the one-component system [7], different analytical forms (which are here linear combinations of hyperbolic functions) should be proposed for X and Y depending on whether $X>X_c$ or $X<X_c$. These forms, as well as their first derivatives, will have to be matched at the spatial location of the transition point, which we have called z_c. Through that matching procedure we get the general solution for the stationary case. A typical form for those solutions is depicted in figure 1. In order to identify the matching point z_c we have to solve $X(z_c)=X_c$. This results in an implicit

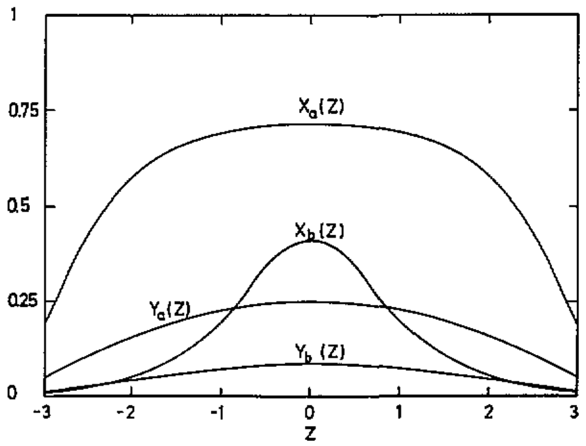

Figure 1. Typical forms for the activator (X) and inhibitor (Y) fields, for the values of the parameters: $X_c=0.3$, $Z_L=3$ and $k=2.5$. The cases indicated with subindex a and b correspond to matching coordinates $Z_c/Z_L=0.92685$ (stable) and $Z_c/Z_L=0.21851$ (unstable), respectively.

equation for z_c and k. In figure 2 we show some general features of the solutions of that implicit equation for different values of X_c. The range of k shown in figure 2 includes positive as well as negative values. Whereas the positive values of k have a clear physical meaning, this does not rule out the possibility of situations which could be described by a negative k (for instance, when there is an active external medium). However, we will focus our discussion on positive values of k. Analogously to the results of [8] we find several branches, which for large values of k tend to the solutions corresponding to Dirichlet boundary conditions. It is worth remarking, as is clearly seen in figure 2, that the Dirichlet limit is attained regardless of the sign of k. For $k=0$, we obtain the solutions corresponding to Neumann boundary conditions. Similarly to the discussion in [8], for intermediate (positive) values of k there appears

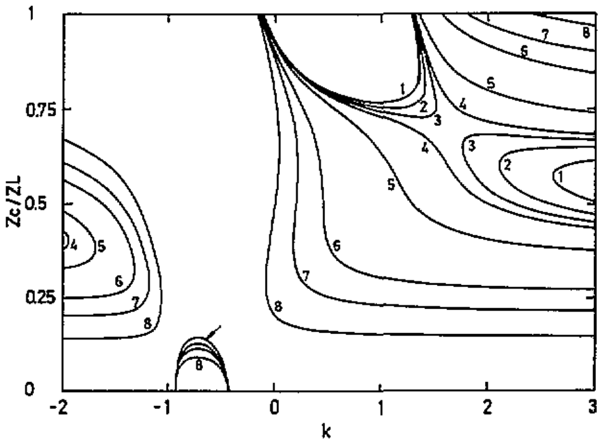

Figure 2. Matching coordinate Z_c in units of Z_L, for positive and negative values of k ($-2<k<3$). Here $Z_L=3$ and the curves correspond to different values of the threshold parameter X_c: (1) $X_c=0.35909$, (2) $..=0.35825$, (3) $..=0.35712$, (4) $..=0.356$, (5) $..=0.35$, (6) $..=0.325$, (7) $..=0.30$, (8) $..=0.25$. The arrow indicates a zone with a coalescence of curves.

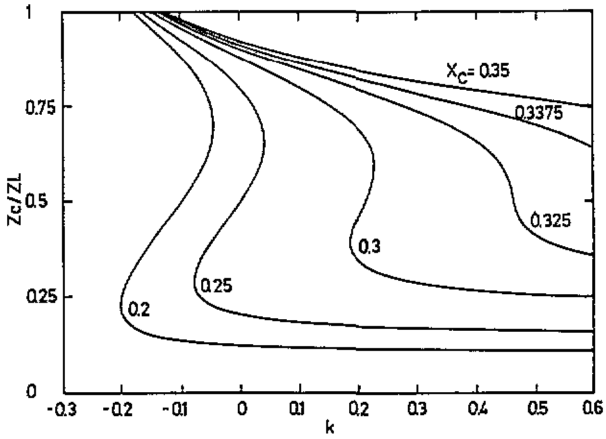

Figure 3. Matching coordinate Z_c in units of Z_L, for an enlargement of figure 2, for $-0.3<k<0.6$. Here also $Z_L=3$, and the values of X_c are indicated in the figure.

an upper and a lower branch, at least for small enough X_c. However, in the present case and due to the interplay of the activator and inhibitor fields, a richer structure arises. For instance, in an enlargement of what is shown in figure 2 for the range $0.1<k<0.9$ we can see that by varying the threshold value X_c of the activator field, a transition occurs from a monotonous behaviour of z_c for X_c large enough, to an S-shaped curve below $X_c=0.32$. This S-like feature recedes towards the region of negative k as X_c becomes smaller. Another very important feature present in figure 2 is shown in detail in figure 4 namely, the formation of a neck due to the coalescence of the upper and lower branches, resembling a fission-like behaviour. Both aspects are new features of the problem under study which were not present in the one-component system previously discussed [8]. it is clear that, when considering different albedo parameters, the complexity of the solutions and the richness of the behaviour in the space (z_c, k_x, k_y) will increase.

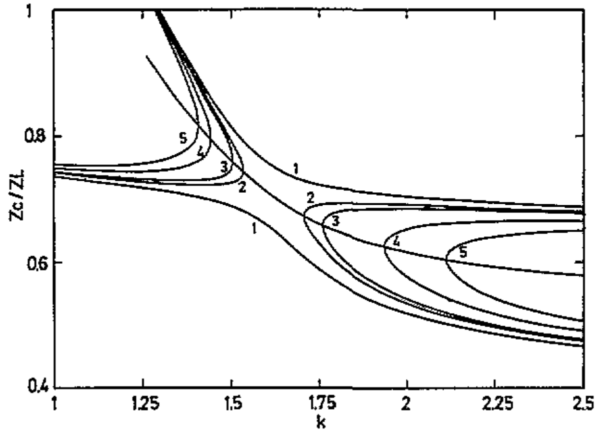

Figure 4. Matching coordinate Z_c in units of Z_L, for $1.0<k<2.5$, with $Z_L=3$, and different values of the threshold parameter X_c: (1) $X_c=0.35656$, (2) $..=0.35700$, (3) $..=0.35712$, (4) $..=0.35768$, (5) $..=0.35825$. The *marginal stability line* is also indicated.

At this point we can ask ourselves about the stability of the solutions so far found. As stated at the beginning, we have done a standard linear stability analysis which was numerically implemented. In contrast with the one-component case, it is far from trivial (since the detailed balance condition is not fulfilled) to find a Lyapunov-like functional describing the global stability of the patterns. Similarly to the findings of [8], the branches lying above the *marginal-stability line* determined by the vertices of the curves as shown in figure 4 correspond to stable stationary solutions whereas those below it are unstable, including the S-shaped branches shown in figure 3. By means of a numerical analysis, we have found that, on the stability line, the pair of values (k, z_c) scale as X_c and $X_c^{1/2}$, respectively. A more detailed analysis of this scaling will be done elsewhere [11].

Aside from the solutions considered here, that is, a symmetric pattern consisting in a central region where the activator field is above a certain threshold $(X>X_c)$ and two lateral regions where it is below it $(X<X_c)$ there are other possibilities, e.g. solutions with alternating regions where $X>X_c$ and $X<X_c$. We expect, according to our experience with the one-component case [8], that this kind of solution will be unstable. However, the analysis of these solutions, as well as the stability of the solutions corresponding to negative values of k, together with the study of the more general case where the albedo parameters are different for each field, will be discussed elsewhere [11]. The interest of the latter case arise from the (realistic) situations where the boundary has different reflectivities for each field (for instance, assume there is a membrane at the boundary, having different porosities for each chemical reactive). A possibility that could arise for given albedo parameters, is that the stationary pattern solution for one of the fields is stable, when considered independently, meanwhile the other is unstable. In such a case we can witness the situation where the unstable pattern is stabilized by the stable one, or vice versa, the stable one is destabilized by the unstable one. The occurrence of such cases is of interest in several physical, chemical and biological systems.

Acknowledgments

Financial support from CONICET through PIA 0523/91, and from Fundación Antorchas is greatly acknowledged. The authors thank Dr V Grunfeld for a critical reading of the manuscript.

References

[1] Hohenberg P C and Cross M C 1986 An introduction to pattern formation in nonequilibrium systems, in *Fluctuations and Stochastic Phenomena in Condensed Matter* ed L Garrido (Berlin: Springer) p 55
Cross P C 1988 Theoretical methods in pattern formation in physics, chemistry and biology in *Far from Equilibrium Phase Transitions* ed L Garrido (Berlin: Springer) p 45

[2] Fife P C 1984 Current topics in reaction-diffusion systems in *Nonequilibrium Cooperative Phenomena in Physics and Related Fields* ed M G Velarde (New York: Plenum) p 371
Nicolis G, Erneux T and Herschkowitz-Kaufman M 1978 Pattern formation in reacting and diffusing system in *Advances in Chemical Physics* vol 38 ed I Prigogine and S Rice (New York: Wiley)
Kuramoto Y 1984 *Chemical Oscillations, Waves and Turbulence* (Berlin: Springer)
Murray J D 1985 *Mathematical Biology* (Berlin: Springer)

[3] Vidal C and Pacault A V 1982 *Evolution of order and chaos in Physics, Chemistry and Biology Synergetics* vol 17 ed H Haken (Berlin: Springer)

[4] Keener J P 1980 *SIAM J. Appl. Math.* **39** 528
[5] Winfree A T 1978 Stable rotating patterns of reaction and diffusion, in *Theoretical Chemistry* vol 4 eds H Eyring and D Henderson (New York: Academic)
[6] Borzi C and Wio H S 1991 *Phys. Lett.* **160A** 247
[7] Kramer L, Ben-Jacob E, Brand H and Cross P C 1982 *Phys. Rev. Lett.* **41** 1891
Kramer L and Hohenberg P C 1984 *Physica* **D13** 357
Hohenberg P C, Kramer L and Riecke H 1985 *Physica* **D15** 402
Ohta T, Ito A and Tetsuka A 1990 *Phys. Rev.* A **42** 3225
[8] Schat C and Wio H S 1992 *Physica* **A180** 295
[9] Kogan B Y, Karplus W J, Billett B S, Pang A T, Karagueuzian H S and Khan S S 1991 *Physica* **D50** 327
Kogan B J, Karplus W J, Karagueuzian H S and Khan S S 1991 Simulation of reentering processes in a heart muscle using a massively parallel computer in *Computers in Biomedicine* eds K D Held, S A Brebbia and R D Ciskowski (Southampton: Computers Mechanical Publications)
[10] Koga S and Kuramoto Y 1980 *Prog. Theor. Phys.* **63** 106
[11] Izus G, Ramirez O, Deza R, Wio H S and Borzi C in preparation